Finiteness Spaces and Groupoids

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The Slogan

Working with finiteness spaces forces certain sums which would a priori be infinite to become finite, and hence well-defined, without resorting to limit structure. This will lead to a great many applications.

This talk contains results from papers with: R. Cockett, J. Beauvais-Feisthauer, I. Dewan, B. Drummond, P.-A. Jacqmin, P. Scott.

Ehrhard's finiteness spaces I

Let X be a set and let $\mathcal{U}$ be a set of subsets of X, i.e., $\mathcal{U} \subseteq \mathcal{P}(X)$. Define $\mathcal{U}^{\perp}$ by:

$$\mathcal{U}^{\perp} = \{u' \subseteq X \mid \text{the set } u' \cap u \text{ is finite for all } u \in \mathcal{U}\}$$

Lemma

- $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \mathcal{V}^{\perp} \subseteq \mathcal{U}^{\perp}$
- $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^{\perp}$

A *finiteness space* is a pair $\mathbb{X} = (X, \mathcal{U})$ with X a set and $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $\mathcal{U}^{\perp\perp} = \mathcal{U}$. We will sometimes denote X by $|\mathbb{X}|$ and $\mathcal{U}$ by $\mathcal{F}(\mathbb{X})$. The elements of $\mathcal{U}$ are called *finitary* subsets.

Finiteness spaces II: Morphisms

- A *morphism* of finiteness spaces $R\colon \mathbb{X} \to \mathbb{Y}$ is a relation $R\colon |\mathbb{X}| \to |\mathbb{Y}|$ such that the following two conditions hold:
  (1) For all $u \in \mathcal{F}(\mathbb{X})$, we have $uR \in \mathcal{F}(\mathbb{Y})$, where $uR = \{y \in |\mathbb{Y}| \mid \exists x \in u, xRy\}$.
  (2) For all $v' \in \mathcal{F}(\mathbb{Y})^{\perp}$, we have $Rv' \in \mathcal{F}(\mathbb{X})^{\perp}$.

Composition is relational and it is straightforward to verify that this is a category. We denote it FinRel.

Lemma (Ehrhard)

In the definition of morphism of finiteness spaces, condition (2) can be replaced with:

(2′) For all $b \in |\mathbb{Y}|$, we have $R\{b\} \in \mathcal{F}(\mathbb{X})^{\perp}$.

Finiteness spaces III: Other choices of morphism

Definition

We define the category FinPf*. Objects are finiteness spaces and a morphism* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ *is a partial function satisfying the same conditions as above.*

Proposition

The category FinPf *is a symmetric monoidal closed, complete and cocomplete category.*

A monoid in this category will be called a *partial finiteness monoid*.

Linearizing finiteness spaces

Let A be an abelian group and $\mathbb{X} = (X, \mathcal{U})$ a finiteness space. Ehrhard defined the abelian group $A\langle\mathbb{X}\rangle$ as the set

$$A\langle\mathbb{X}\rangle = \{f : X \to A \,|\, supp(f) \in \mathcal{U}\}$$

together with pointwise addition, where:
$supp(f) = \{x \in X | f(x) \neq 0\}$.

In a previous paper, we showed that Ribenboim's construction of generalized polynomial rings is an instance of finding finiteness space structures on certain partially ordered monoids.

Linearizing II

Theorem

If $(\mathbb{M}, \mu\colon \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}, \eta\colon I \to \mathbb{M})$ *is a partial finiteness monoid and* R *a ring (not necessarily commutative, but with unit), then* $R\langle\mathbb{M}\rangle$ *canonically has the structure of a ring.*
The multiplication in $R\langle\mathbb{M}\rangle$ *is given by*

$$(f \cdot g)(m) = \sum_{(m_1,m_2) \in X_m(f,g)} f(m_1) \cdot g(m_2).$$

where:
$X_m(f, g) = \{(m_1, m_2) \,|\, m_1 + m_2 = m,\ f(m_1) \neq 0,\ g(m_2) \neq 0\}$

We can replace ring with semiring or even ordered ring or ordered semiring and the above still works.

The sum is finite.

Why is the set $X_m(f,g)$ finite?

This set is exactly

$$\underbrace{(\text{supp}(f) \times \text{supp}(g))}_{\in \mathcal{W}} \cap \underbrace{\mu^{-1}(m)}_{\in \mathcal{W}^{\perp}}$$

Recall that μ is the multiplication. $\mathcal{W}$ is the finiteness space structure for $\mathbb{M} \otimes \mathbb{M}$.

Definition (For Category Theorists)

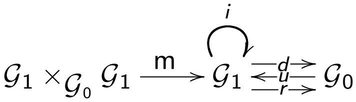
A *groupoid* is a (small) category in which every morphism is invertible.

Definition (For Functional Analysts)

A *groupoid* is a pair of sets $\mathcal{G}_1$ (arrows) and $\mathcal{G}_0$ (objects) with morphisms

- $d, r \colon \mathcal{G}_1 \to \mathcal{G}_0$
- $m \colon \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$
- $u \colon \mathcal{G}_0 \to \mathcal{G}_1 \qquad i \colon \mathcal{G}_1 \to \mathcal{G}_1$

satisfying evident axioms.



Examples of groupoids

- Any group is a one-object groupoid.
- Any disjoint union of groups. This is called a *group bundle*.
- The fundamental groupoid of a space.
- An equivalence relation induces a groupoid where there is precisely one arrow between two elements if they are equivalent.
- A group action induces a groupoid:
  Let G act on a set X. Let $\mathcal{G}_1 = G \times X$ and $\mathcal{G}_0 = \{e\} \times X$. Then

$$d(g,x) = x, \quad r(g,x) = gx, \quad (g,hx)\cdot(h,x) = (gh,x)$$

Internal groupoids

Given a category with finite limits, one can consider groupoids internal to that category, since the definition can be expressed entirely diagrammatically. A *localic groupoid* is a groupoid internal to the category of locales.

Theorem (Joyal-Tierney)

Every Grothendieck topos is equivalent to a category of sheaves on a localic groupoid.

Theorem (Moerdijk)

The above extends to an equivalence of 2-categories.

Topological groupoids

These can be defined with various levels of generality. We'll follow A. Sims, *Hausdorff étale groupoids and their C^*-algebras*

- A *topological groupoid* is a groupoid $\mathcal{G}$ in the category of locally compact hausdorff spaces and continuous maps.

- A topological groupoid is *étale* if its domain map is a local homeomorphism. (This implies the range map and multiplication are as well.)

Associating algebras to étale groupoids

Theorem

Let $\mathcal{G}$ be a second countable étale groupoid. Let

$$\mathcal{C}_c(\mathcal{G}) = \{f : \mathcal{G} \to \mathbb{C} | supp(f) \text{ is compact}\}$$

Define $f \star g : \mathcal{G} \to \mathbb{C}$ by

$$f \star g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$$

Then $\mathcal{C}_c(\mathcal{G})$ is a $*$-algebra with above multiplication and $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

The key point is that this sum is finite. There is a purely topological argument to support this. For us, the finiteness of the sum follows from properties of finiteness spaces.

Topological spaces as finiteness spaces?

To carry out the argument mentioned above, it makes sense whether to ask if there is a class of sufficiently nice topological spaces X such that $(X, \mathcal{U})$ is a finiteness space where $\mathcal{U}$ is the set of relatively compact subsets and $\mathcal{U}^{\perp}$ is the set of discrete, closed subspaces. (A subspace is *relatively compact* if its closure is compact in X.)

For general topological spaces, this is certainly false. But a reasonable conjecture is that the results hold for locally compact hausdorff spaces.

Topological spaces as finiteness spaces I

The following is the work of Joey Beauvais-Feisthauer, Ian Dewan & Blair Drummond.

Theorem

The conjecture is false. The smallest uncountable ordinal ω_1, with the order topology, is locally compact and Hausdorff but not a finiteness space under the above structure.

But a smaller class of spaces does work.

Definition

- *X is σ-compact if it can be covered by a countable family of compact subsets.*
- *X is σ-locally compact if it is both σ-compact and locally compact.*

Topological spaces as finiteness spaces II

Theorem (B-F,D,D)

- *Let X be a σ-locally compact hausdorff space. Then it is a finiteness space.*
- *The converse is false. Let X be an uncountable discrete space. Then X is locally compact and hausdorff, but not σ-compact. But X is a finiteness space.*

Nonetheless, the class of σ-locally compact Hausdorff spaces is quite large, e.g. it contains manifolds and every CW-complex with countably many cells.

Some of our étale groupoids have underlying spaces which are σ-locally compact Hausdorff. We will use this fact to give a new approach to constructing algebras for them.

The following is due to Kumjian, Pask, Raeburn and Renault. We'll call this the *KPRR-groupoid*. See the paper:

Graphs, groupoids and Cuntz-Krieger algebras, by the above authors. They show that the C*-algebras of this form are of fundamental importance. One of their theorems, stated somewhat imprecisely.

Theorem

The C-algebra attached to a graph of the above form is the universal C*-algebra generated by (possibly infinite) families of partial isometries satisfying Cuntz-Krieger relations determined by the graph.*

A finiteness topological groupoid II

Let $G = (V, E)$ be a directed graph with V countable. We'll also assume G is row-finite, i.e. for all vertices v, $d^{-1}(v)$ is finite. Let $P(G)$ be the set of all infinite paths and $F(G)$ be the set of all finite paths. $P(G)$ can be seen as a subspace:

$$P(G) \subseteq \prod_{i=1}^{\infty} E \quad \text{with } E \text{ topologized discretely}$$

The topology can be described as follows. If $\alpha \in F(G)$, let

$$Z(\alpha) = \{x \in P(G) | x = \alpha y, \quad \text{with } y \in P(G)\}$$

Theorem (KPRR)

The sets $\{Z(\alpha) | \alpha \in F(G)\}$ form a basis for the topology on $P(G)$. The resulting topology is locally compact and totally disconnected.

A finiteness topological groupoid III

$P(G)$ is the object part of an étale groupoid.

Definition

Suppose $x, y \in P(G)$. We say that x and y are *shift equivalent with lag* $k \in \mathbb{Z}$ if there exists $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i > N$. We write $x \sim_k y$.

Lemma

We have $x \sim_0 x$ *and* $x \sim_k y \Rightarrow y \sim_{-k} x$ *and* $x \sim_k y, y \sim_l z \Rightarrow x \sim_{k+l} z$.

Define

$$\mathcal{G} = \{(x, k, y) \in P(G) \times \mathbb{Z} \times P(G) | x \sim_k y\}$$

A finiteness topological groupoid IV

Define a multiplication $\mu \colon \mathcal{G}^2 \to \mathcal{G}$

$$\mu((x,k,y_1)(y_2,l,z)) \mapsto$$
$$\begin{cases} \text{undefined} & \text{if } y_1 \neq y_2 \\ (x, k+l, z) & \text{if } y_1 = y_2 \end{cases}$$

with inverse given by $i(x,k,y) = (y,-k,x)$

Theorem

Let G be a row-finite directed graph. The sets

$$\{Z(\alpha,\beta) | \alpha,\beta \in F(G) \text{ and } r(\alpha) = r(\beta)\}$$

form a basis for a locally compact Hausdorff topology on $\mathcal{G}$. With this topology, $\mathcal{G}$ is a second countable, locally compact and σ-locally compact étale groupoid.

It works

Theorem

$\mathcal{G}$ with the relatively compact structure is a finiteness space with the relatively compact structure. The multiplication of the groupoid makes this a partial finiteness monoid. So linearization gives us a K-algebra. If K has a $*$-operation, then the result is a $*$-algebra.

But at the moment, our construction is discrete. In the groupoid approach to C*-algebras, after building the $*$-algebra, one defines a norm and then completes with respect to the norm to obtain a C*-algebra. Can we add topology in our construction? Yes, but this is quite different than the usual topology of functional analysis.

Pause: Where did we use the étale property?

Our finiteness space construction on topological spaces is not functorial if we allow arbitrary continuous maps. If the map identifies too many points, we could lose the finiteness property.

Definition

Let $f : X \longrightarrow Y$ be a partial function between two topological spaces. We say that f is *locally finite-to-one* when the (total) function $f : \mathrm{dom(f)} \longrightarrow Y$ is locally finite-to-one, i.e., for each $x \in \mathrm{dom(f)}(f)$, there exists a neighbourhood U of x such that the restriction map $f_{|_U}$ has no infinite fibres.

Proposition

The finiteness construction is functorial when we restrict to only continuous locally finite-to-one maps.

Evidently, étale maps are locally finite to one.

Lefschetz topology

Lefschetz introduced these topological spaces to improve the dualities of vector spaces. If V^* denotes the dual space of V, it is standard that the canonical embedding $V \to V^{**}$ is an isomorphism if and only if V is finite-dimensional. But by introducing topology and redefining V^* to be the linear *continuous* maps, then we can perhaps reduce the size of V^{**} by just the right amount.

Definition

A vector space is a *Lefschetz space* if equipped with a T_0-topology such that

- The vector operations are continuous, i.e. it is a topological vector space. (We'll assume that the base field is discrete.)
- $0 \in V$ has a neighborhood basis of open linear subspaces.

The category of Lefschetz spaces and continuous linear maps will be denoted Lef.

Lefschetz topology II

Lemma (Barr)

Lef *is symmetric, monoidal closed. The tensor is described by a topology on the algebraic tensor product.*

Lemma (Lefschetz)

The embedding $\rho\colon V \to V^{**}$ is a bijection for all Lefschetz spaces.

Theorem (Barr)

The full subcategory of those spaces for which ρ is an isomorphism is $*$-autonomous.

Lefschetz topology III

As observed by Ehrhard, there is a topology one can place on the linearizations of finiteness spaces.

Definition

Let $(X, \mathcal{U})$ be a finiteness space. Let $u' \in \mathcal{U}^{\perp}$. Let

$$V_{u'} = \{f \in R\langle X\rangle | supp(f) \cap u' = \emptyset\}$$

This determines a neighborhood basis at the point $0 \in R\langle X\rangle$. The resulting topology is a Lefschetz topology.

Lefschetz topology IV

Here are some properties, as observed by Ehrhard, for a finiteness space $(X, \mathcal{U})$.

Lemma

- *If $\mathcal{U}$ consists of just the finite subsets, then $R\langle X\rangle$ gets the discrete topology.*
- *If $\mathcal{U} = \mathcal{P}(X)$, then $R\langle X\rangle = R^X$ with the product topology*
- *If $f_n \in R\langle X\rangle$ for all $n \in \mathbb{N}$, then $\sum_0^\infty f_n$ converges if and only if $\lim_n f_n \to 0$.*

This topology is very different from the C^* one discussed earlier.

- What precisely is the relationship?
- Can the computational properties of linear topology considered by Ehrhard and Tasson say anything about the corresponding C^*-algebras?

Next?

We continue to look for applications of finiteness spaces. We're especially interested in *monoidal topology*, as described in

- D. Hofmann, G. Seal, W. Tholen (editors). **Monoidal Topology: A Categorical Approach to Order, Metric and Topology**, (2013).

There we consider *quantale-valued relations*, i.e. functions $R\colon X \times Y \to Q$, which compose via the formula:

$$R; S(x, z) = \bigvee_{y \in Y} R(x, y) \otimes S(y, z)$$

Many interesting geometric structures arise naturally as *Q-categories*, such as metric spaces, topological spaces and approach spaces.

Replacing sets with finiteness spaces allows us to replace quantales with partially ordered semirings.

Next?

What sort of structures arise from this shift? I'll tell you all about this at....

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Thomas's 65th birthday!