Multicoherences

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Workshop in Honour of Thomas Ehrhard's 60th Birthday

Outline

Twenty years old story

Power coherence spaces

Definition

A power is an endofunctor of the category of sets and inclusions.

Definition

Let P be a power, a P-coherence space X is a triple $(|X|, \frown_X, \asymp_X)$, $|X|$ the web at most countable, $\frown_X, \asymp_X \subseteq P(|X|)$ and $\frown_X \cup \asymp_X = P(|X|)$.

Examples

- $\emptyset$-coherence space : relational semantics, **Rel**
- $\mathcal{P}^*_{\mathrm{fin}}$ (finite non-empty sets) : hypercoherences, **NHc**
- $\mathcal{M}_{\{2\}}$ power = pairs : coherence spaces, $\mathbf{NCoh}_{\{2\}}$
- $\mathcal{M}_{\mathbb{N}\setminus\{0,1\}}$ = finite multisets $\# \geq 2$, multicoherences **NMc**

Coherence

Definition

Let X be a P-coherence space. In $P(|X|)$, $\frown_X$ is the coherence relation, $\asymp_X$ is the incoherence relation, $P(|X|) \setminus \frown_X$ the strict incoherence relation, $P(|X|) \setminus \asymp_X$ the strict coherence relation and $\frown_X \cap \asymp_X$ is the neutral relation, denoted N_X.

Definition

A clique of X (a P-coherence space) is a $x \subseteq |X|$ s.t. $P(x) \subseteq \frown_X$.

The linear negation exchanges $\frown_X$ and $\asymp_X$.

Definition

X is deterministic if for each clique x of X, each clique y of $X^\perp$, $\#(x \cap y) \leq 1$.

Uniformity

Definition

(Informal) an expectation to interact with a fair value.

Example

$$\begin{aligned} M = \lambda b.\ \ \texttt{if}\ b\ \ & \texttt{then}\ \{\texttt{if}\ b\ \texttt{then}\ M_1\ \texttt{else}\ M_2\} \\ & \texttt{else}\ \{\texttt{if}\ b\ \texttt{then}\ M_3\ \texttt{else}\ M_4\} \end{aligned}$$

- Multiset-based exponentials:

$$\{([\mathrm{t},\mathrm{t}], v_1), ([\mathrm{t},\mathrm{f}], v_2), ([\mathrm{t},\mathrm{f}], v_3), ([\mathrm{f},\mathrm{f}], v_4)\}$$

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- Set-based exponentials: $\{(\{\mathrm{t}\}, v_1), (\{\mathrm{f}\}, v_4)\}$

Neutral web

Property

NCoh, **NHc**, **NMc** *enjoy a stronger*[1] *property than determinism :* $\mathsf{N}_X \subseteq \cup_{a \in |X|} P(\{a\})$.

Definitions

The neutral web $|X|_{\mathsf{N}}$ is the set $\{a \in |X| \mid P(\{a\}) \subseteq \mathsf{N}_X\}$. The neutral restriction of X is the sub-space of X of web $|X|_{\mathsf{N}}$, that is $(|X|_{\mathsf{N}}, \frown_X \cap M, \asymp_X \cap M)$ where $M = P(|X|_{\mathsf{N}})$, and the neutral restriction of a clique x of X is $x \cap |X|_{\mathsf{N}}$.

The space is reduced to the expected sites of interactions.

[1] provided that the power is strictly monotone and preserves disjointness

Extensional collapse

Definition

The extensional collapse of a semantics, with respect to the simple types hierarchy, is its quotient by the partial equivalence relations $\sim_A$, given by equality on basis types and :

$$f \sim_{A \Rightarrow B} g \text{ iff if } x \sim_A y \text{ then } f(x) \sim_B g(y).$$

Theorem

The extensional collapse of **NCoh** *and of the multiset-based* **Coh** *semantics is the set-based* **Coh** *semantics. The same for* **NHc**/**Hc**, **NMc**/**Mc**.

The neutral restiction plays a crucial role in the proof (all the commutations one can expects).

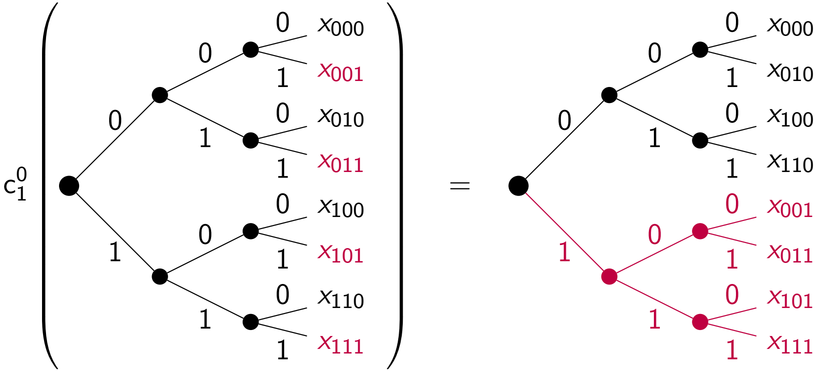
Accuracy

- No full completeness (too simple?)
- Gustave function in coherence spaces
- In **Hc**, **Mc** subdefinability: for any clique x of $\mathbf{Bool}^n \to \mathbf{Bool}$ there exists a term M s.t. $x \subseteq [M]$.
- **Hc** extensional collapse of the sequential algorithms
- non-uniform is non-sequential. $\{([\mathrm{t}], \mathrm{t}), ([\mathrm{f}], \mathrm{t}), ([\mathrm{t}, \mathrm{f}], \mathrm{t})\}$

Coherent Differential PCF syntax

- Thomas Ehrhard, A coherent differential PCF. 2022
- Differential lambda-calculus without the general sum $M + N$
- **NCoh**, **NHc**, **NMc** are valid semantics
- types: $A, B := D^n\iota \mid A \Rightarrow B$
- D applies to any type but $D(A \Rightarrow B) = A \Rightarrow DB$
- D applies to terms and triggers a rewriting $D(\lambda x^A.M : A \Rightarrow B) = \lambda.x^{DA}\partial(x, M) : DA \Rightarrow DB$
- $|D^n\iota| = \{(\delta, i) \mid i \in \mathbb{N}, \delta$ word of n bits$\}$
- projections π_0, π_1; injections $\mathrm{in}_0, \mathrm{in}_1$; swap c; delayed sum θ
- $\mathrm{succ}^d, \mathrm{if}^d_A, \mathrm{let}^d_A, \pi^d_0, \theta^d \ldots$ at depth d
- Don't confuse $D^0\iota = \iota$ but $\pi^0_i(M) = \pi_i(M)$.

Swap


c_1^0
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Rel intersection typing system (examples)

$$\dfrac{\dfrac{\pi_0 \qquad \dfrac{\pi_1 \qquad \dfrac{\vdots}{x : \iota \vdash 2 : \iota}\,\text{typing} \qquad \dfrac{}{x : [] : \iota \vdash 3 : \iota}\,\text{num}}{x : [1] : \iota \vdash \mathrm{if}_\iota(x, 2, 3) : 3 : \iota}\,\text{if1} \qquad \pi_2}{x : [0,1] : \iota \vdash \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5)) : 3 : \iota}\,\text{if0}}{\vdash \lambda x^\iota.\, \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5)) : ([0,1], 3) : \iota \Rightarrow \iota}\,\text{abs}$$

Where :

$$\pi_0 = \dfrac{}{x : [0] : \iota \vdash x : 0 : \iota}\,\text{var} \qquad \pi_1 = \dfrac{}{x : [1] : \iota \vdash x : 1 : \iota}\,\text{var}$$

$$\pi_2 = \dfrac{\vdots}{x : \iota \vdash \mathrm{if}_\iota(x, 4, 5) : \iota}\,\text{typing}$$

Example 2, $x : D\iota$

$$\dfrac{\pi_0 \qquad \dfrac{\pi_1 \qquad \pi_3 \qquad \dfrac{}{x : [] : D\iota \vdash 3 : \iota}\,\text{num}}{x : [(\delta', 1)] : D\iota \vdash \text{if}^1_\iota(x, 2, 3) : (\delta', 3) : D\iota}\,\text{if1} \qquad \pi_2}{\dfrac{x : [(\delta, 0), (\delta', 1)] : D\iota \vdash \text{if}^1_\iota(x, \text{if}^1_\iota(x, 2, 3), \text{if}^1_\iota(x, 4, 5)) : (\delta\delta', 3) : D^2\iota}{\vdash \lambda x^\iota . \text{if}^1_{D\iota}(x, \text{if}^1_\iota(x, 2, 3), \text{if}^1_\iota(x, 4, 5)) : ([(\delta, 0), (\delta', 1)], (\delta\delta', 3)) : D\iota \Rightarrow D^2\iota}\,\text{abs}}\,\text{if0}$$

Where :

$$\pi_0 = \dfrac{}{x : [(\delta, 0)] : D\iota \vdash x : (\delta, 0) : D\iota}\,\text{var}$$

$$\pi_1 = \dfrac{}{x : [(\delta', 1)] : D\iota \vdash x : (\delta', 1) : D\iota}\,\text{var}$$

$$\pi_2 = \dfrac{\vdots}{x : D\iota \vdash \text{if}^1_\iota(x, 4, 5) : D\iota}\,\text{typing} \qquad \pi_3 = \dfrac{\vdots}{x : D\iota \vdash 2 : \iota}\,\text{typing}$$

Example 3, differentiation

$$\dfrac{\dfrac{\pi_0 \qquad \dfrac{\pi_1 \qquad \pi_3 \qquad \dfrac{}{x : [\,] : \iota \vdash 1 : \iota}\,\text{num}}{x : [\mathrm{f}] : \iota \vdash \mathrm{if}_\iota(x, 2, 3) : 3 : \iota}\,\text{if1} \qquad \pi_2}{x : [\mathrm{t}, \mathrm{f}] : \iota \vdash \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5)) : 3 : \iota}\,\text{if0}}{\dfrac{\vdash \lambda x^\iota.\, \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5)) : ([\mathrm{t}, \mathrm{f}], 3) : \iota \Rightarrow \iota}{}\,\text{abs} \qquad a \in \partial_{[\![\iota]\!]}}$$

$$\dfrac{\vdash \lambda x^\iota.\, \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5)) : ([\mathrm{t}, \mathrm{f}], 3) : \iota \Rightarrow \iota \qquad a \in \partial_{[\![\iota]\!]}}{\vdash D(\lambda x^\iota.\, \mathrm{if}_\iota(x, \mathrm{if}_\iota(x, 2, 3), \mathrm{if}_\iota(x, 4, 5))) : ([(0, \mathrm{t}), (1, \mathrm{f})], (1, 3)) : \iota \Rightarrow \iota}\,\text{map}$$

Where $a = ([(0, \mathrm{t}), (1, \mathrm{f})], (1, [\mathrm{t}, \mathrm{f}]))$ and :

$$\pi_0 = \dfrac{}{x : [\mathrm{t}] : \iota \vdash x : \mathrm{t} : \iota}\,\text{var} \qquad \pi_1 = \dfrac{}{x : [\mathrm{f}] : \iota \vdash x : \mathrm{f} : \iota}\,\text{var}$$

$$\pi_2 = \dfrac{\vdots}{x : \iota \vdash \mathrm{if}_\iota(x, 4, 5) : \iota}\,\text{typing} \qquad \pi_3 = \dfrac{\vdots}{x : \iota \vdash 2 : \iota}\,\text{typing}$$

Conclusion

- Uniformity appeared to be a kind of anticipation of the sites of interaction
- This intuition works pretty well for the closed case where a clique encounter an anti-clique, as for the extensional collapses.
- I am still digging to understand coherent differential PCF.
- This calculus admits **NCoh**, **NHc**, **NMc** but ~~not their uniform versions~~[2].
- What is wrong? Is there an answer in terms of interaction?

[2] That was a mistake in my talk, as confirmed by T. Ehrhard, uniform coherent spaces are a model of Coh. Diff. PCF

Conclusion

Thank you!