The Cartesian Closed Bicategory of Cartesian Distributors

Marcelo Fiore
University of Cambridge

Thomas Ehrhard's 60^{th} Birthday Workshop

29.IX.2022

- Joint work with:
  André Joyal
  Université du Québec à Montréal

- With many thanks to the workshop organisers:
  Antonio Bucciarelli, Michele Pagani, Laurent Regnier
  Christine Tasson, Lorenzo Tortora de Falco,
  Lionel Vaux Auclair

PLAN

- Present
  The cartesian closed bicategory of cartesian distributors
- Discuss it in
  connection to linear logic
- Outline the
  enveloping mathematical theory

The Cartesian Closed Bicategory of Cartesian Distributors

<u>**CartDist**</u>

objects: (unsorted) algebraic theories $\left(=^{\mathrm{def}} \text{small cartesian categories} \right)$

morphisms: $\mathbb{A} \longleftrightarrow \mathbb{B}$

$=^{\mathrm{def}}$ right cartesian $\mathbb{A}^{\circ} \times \mathbb{B} \longrightarrow \underline{\underline{\mathrm{Set}}}$

cells: natural transformations

The Cartesian Closed Bicategory of Cartesian Distributors

$\underline{\underline{\text{CartDist}}}$

objects: (unsorted) algebraic theories $\left(=^{\text{def}} \begin{array}{l} \text{small} \\ \text{cartesian} \\ \text{categories} \end{array} \right)$

morphisms: $\mathbb{A} \longleftrightarrow \mathbb{B}$

$=^{\text{def}}$ right cartesian $\mathbb{A}^{\circ} \times \mathbb{B} \longrightarrow \underline{\underline{\mathrm{Set}}}$

cells: natural transformations

$\underline{\underline{\text{Thm}}}$ [Fiore & Joyal (2015)]

$\underline{\underline{\text{CartDist}}}$ is a cartesian closed bicategory

Generalizations

The results herein extend

- to the enrich context

and

- to essentially algebraic theories (small categories with finite limits)

Cartesian Categories and Functors

$\underline{\underline{\text{CART}}} \longleftarrow\!\!\rightarrow \underline{\underline{\text{AlgTh}}}$

objects: cartesian categories

morphisms: cartesian functors

cells: natural transformations

$\underline{\underline{\text{NB}}}$: For $f : \mathbb{A} \to \mathbb{B}$ in $\underline{\underline{\text{AlgTh}}}$,

$$f_! \dashv f^* : \mathbb{B} \nrightarrow \mathbb{A} \text{ in } \underline{\underline{\text{CartDist}}}$$

with

$$f_!(a,b) = \mathbb{B}(f(a), b) \quad , \quad f^*(b,a) = \mathbb{B}(b, f(a))$$

Cartesian Categories and Functors

$$\underline{\underline{\mathrm{CART}}} \hookleftarrow \underline{\underline{\mathrm{AlgTh}}}$$

objects: cartesian categories
morphisms: cartesian functors
cells: natural transformations

- Biproducts

$$\mathbb{1} \rightleftarrows \mathcal{A}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "{(-,1)}"'] & \mathcal{A}\times\mathcal{B} \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & \mathcal{B} \arrow[l, shift left, "{(1,-)}"] \end{tikzcd}$$

- Closed structure

hom $\underline{\underline{\mathrm{CART}}}(\mathcal{A}, \mathcal{B})$ unit $\mathbb{F} = \underline{\underline{\mathrm{FinSet}}}^{op}$

$\left(\underline{\underline{\mathrm{NB}}}: \ \underline{\underline{\mathrm{CART}}}(\mathcal{A}, \mathcal{B}) \times \mathcal{A} \xrightarrow{ev} \mathcal{B} \text{ is left-and-right cartesian}\right)$

- Closed structure

hom $\underline{\underline{\mathrm{CART}}}(\mathbb{A}, \mathbb{B})$ unit $\mathbb{F} = \underline{\underline{\mathrm{FinSet}}}^{op}$

(NB: $\underline{\underline{\mathrm{CART}}}(\mathbb{A},\mathbb{B}) \times \mathbb{A} \xrightarrow{ev} \mathbb{B}$ is left-and-right cartesian)

- Symmetric monoidal closed structure

left-and-right cartesian classifier

$\mathscr{A} \times \mathbb{B} \longrightarrow \mathscr{A} \otimes \mathbb{B}$ left-and-right cartesian

$\underline{\underline{\mathrm{CART}}}(\mathscr{A} \otimes \mathbb{B}, \mathbb{C}) \simeq \underline{\underline{\mathrm{lrCART}}}(\mathscr{A} \times \mathbb{B}, \mathbb{C})$

- Tensors and powers [Fiore & Joyal (2015), Fiore & Joyal (2020)]

$$\underline{\underline{\mathrm{CART}}}(\mathbb{I} * \mathcal{A}, \mathcal{B}) \cong \underline{\underline{\mathrm{CAT}}}(\mathbb{I}, \underline{\underline{\mathrm{CART}}}(\mathcal{A}, \mathcal{B})) \cong \underline{\underline{\mathrm{CART}}}(\mathcal{A}, \mathcal{B}^{\mathbb{I}})$$

▶ Tensors and powers

$$
\begin{tikzcd}[column sep=tiny]
 & \underline{\underline{\mathrm{CAT}}}\left(\mathbb{I}, \underline{\underline{\mathrm{CART}}}(\mathcal{A}, \mathcal{B})\right) \arrow[dl, phantom, "\simeq" sloped] \arrow[dr, phantom, "\simeq" sloped] & \\
\underline{\underline{\mathrm{CART}}}\left(\mathbb{I} * \mathcal{A}, \mathcal{B}\right) & & \underline{\underline{\mathrm{CART}}}\left(\mathcal{A}, \mathcal{B}^{\mathbb{I}}\right)
\end{tikzcd}
$$

NB: left-and-right closed tensor action

$$\underline{\underline{\mathrm{CAT}}} \times \underline{\underline{\mathrm{CART}}} \xrightarrow{\;*\;} \underline{\underline{\mathrm{CART}}}$$

$$\mathbf{0} * \mathcal{A} \simeq \mathbf{0} \qquad (\mathbb{I} + \mathbb{J}) * \mathcal{A} \simeq (\mathbb{I} * \mathcal{A}) \times (\mathbb{J} * \mathcal{A})$$

$$\mathbb{I} * \mathbf{1} \simeq \mathbf{1} \qquad \mathbb{I} * (\mathcal{A} \times \mathcal{B}) \simeq (\mathbb{I} * \mathcal{A}) \times (\mathbb{I} * \mathcal{B})$$

$$\mathbf{1} * \mathcal{A} \simeq \mathcal{A} \qquad \mathbb{J} * (\mathbb{I} * \mathcal{A}) \simeq (\mathbb{J} \times \mathbb{I}) * \mathcal{A}$$

Formulas for Free Cartesian Categories

$$\begin{tikzcd} \underline{\underline{\mathrm{CART}}} \arrow[d, shift left=3] \\ \underline{\underline{\mathrm{CAT}}} \arrow[u, bend left=40, shift left=3] \arrow[u, phantom, "\dashv"] \arrow[loop left, "F"] \end{tikzcd}$$

cartesian completion

Formulas for Free Cartesian Categories

$$
\begin{tikzcd}
\underline{\underline{\mathrm{CART}}} \arrow[d, bend left] \\
\underline{\underline{\mathrm{CAT}}} \arrow[u, bend left] \arrow[u, phantom, "\dashv"] \arrow[loop left, "\mathcal{F}"]
\end{tikzcd}
$$

cartesian completion

$$
\begin{cases}
\mathcal{F}(\mathbf{0}) \simeq \mathbf{1} \\
\mathcal{F}(I+J) \simeq \mathcal{F}(I) \times \mathcal{F}(J)
\end{cases}
$$

Formulas for Free Cartesian Categories

$$\begin{tikzcd} \underline{\underline{\mathrm{CART}}} \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\ \underline{\mathrm{CAT}} \arrow[u, shift left=2] \arrow[loop left, "\mathcal{F}"] \end{tikzcd}$$

cartesian completion

$$\begin{cases} \mathcal{F}(\mathbf{0}) \simeq \mathbf{1} \\ \mathcal{F}(I+J) \simeq \mathcal{F}(I) \times \mathcal{F}(J) \end{cases}$$

$$\begin{cases} \mathbf{F} \simeq \mathcal{F}(1) \\ \mathcal{F}(I) \otimes \mathcal{F}(J) \simeq \mathcal{F}(I \times J) \end{cases}$$

Formulas for Free Cartesian Categories

$$
\begin{tikzcd}
\underline{\underline{\mathrm{CART}}} \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\
\underline{\underline{\mathrm{CAT}}} \arrow[u, bend left] \arrow[loop left, "\mathcal{F}"]
\end{tikzcd}
$$

cartesian completion

$$
\begin{cases}
\mathcal{F}(\mathbf{0}) \simeq \mathbf{1} \\
\mathcal{F}(\mathcal{I}+\mathcal{J}) \simeq \mathcal{F}(\mathcal{I}) \times \mathcal{F}(\mathcal{J})
\end{cases}
$$

$$
\begin{cases}
\mathbf{F} \simeq \mathcal{F}(1) \\
\mathcal{F}(\mathcal{I}) \otimes \mathcal{F}(\mathcal{J}) \simeq \mathcal{F}(\mathcal{I} \times \mathcal{J})
\end{cases}
$$

$$
\begin{cases}
\mathcal{I} * \mathcal{A} \simeq \mathcal{F}(\mathcal{I}) \otimes \mathcal{A} \\
\mathcal{I} * \mathcal{F}(\mathcal{J}) \simeq \mathcal{F}(\mathcal{I} \times \mathcal{J}) \\
\mathcal{A}^{\mathbb{I}} \simeq \underline{\underline{\mathrm{CART}}}(\mathcal{F}(\mathbb{I}), \mathcal{A})
\end{cases}
$$

Cartesian distributors revisited

$$
\begin{array}{c}
\mathbb{A} \xrightarrow{\quad C \quad} \mathbb{B} \\
\hline\hline
\mathbb{A}^{o} \times \mathbb{B} \xrightarrow[\substack{\text{right} \\ \text{cartesian}}]{\qquad\qquad} \underline{\underline{\mathrm{Set}}} \\
\hline\hline
\mathbb{A}^{o} * \mathbb{B} \xrightarrow[\text{cartesian}]{\qquad\qquad} \underline{\underline{\mathrm{Set}}}
\end{array}
$$

Cartesian structure of <u>CartDist</u>

induced by the left-and-right closed tensor action

Cartesian structure of $\underline{\underline{\text{CartDist}}}$

induced by the left-and-right closed tensor action

- $$\frac{\mathbb{A} \hookrightarrow \mathbb{1}}{\mathbb{1} \simeq \mathbb{A}^\circ * \mathbb{1} \xrightarrow[\text{cartesian}]{} \underline{\underline{\text{Set}}}}$$

Cartesian structure of $\underline{\underline{\mathrm{CartDist}}}$

induced by the left-and-right closed tensor action

- $$\frac{\mathbb{A} \xrightarrow{\;\;} \mathbb{1}}{\mathbb{1} \simeq \mathbb{A}^\circ * \mathbb{1} \xrightarrow[\text{cartesian}]{} \underline{\underline{\mathrm{Set}}}}$$

- $$\frac{\dfrac{\mathbb{C} \xrightarrow{\;\;} \mathbb{A} \times \mathbb{B}}{(\mathbb{C}^\circ * \mathbb{A}) \times (\mathbb{C}^\circ * \mathbb{B}) \simeq \mathbb{C}^\circ * (\mathbb{A} \times \mathbb{B}) \xrightarrow[\text{cartesian}]{} \underline{\underline{\mathrm{Set}}}}}{\dfrac{(\mathbb{C}^\circ * \mathbb{A}) \xrightarrow[\text{cartesian}]{} \underline{\underline{\mathrm{Set}}}}{\mathbb{C} \xrightarrow{\;\;} \mathbb{A}} \qquad \dfrac{(\mathbb{C}^\circ * \mathbb{B}) \xrightarrow[\text{cartesian}]{} \underline{\underline{\mathrm{Set}}}}{\mathbb{C} \xrightarrow{\;\;} \mathbb{B}}}$$

Cartesian Closed structure of CartDist

$$\mathbb{A} \times \mathbb{B} \xrightarrow{\;c\;} \mathbb{C}$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$\mathbb{A}^\circ * (\mathbb{B}^\circ * \mathbb{C}) \simeq (\mathbb{A}^\circ \times \mathbb{B}^\circ) * \mathbb{C} \cong (\mathbb{A} \times \mathbb{B})^\circ * \mathbb{C} \xrightarrow[\text{cartesian}]{} \underline{\underline{\mathrm{Set}}}$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$\mathbb{A} \xrightarrow{\;c\;} \mathbb{B}^\circ * \mathbb{C}$$

Cartesian closed structure of $\underline{\underline{\text{CartDist}}}$

- $$A \times B \longrightarrow \mathbb{C}$$

$$A^\circ * (B^\circ * \mathbb{C}) \simeq (A^\circ \times B^\circ) * \mathbb{C} \cong (A \times B)^\circ * \mathbb{C} \xrightarrow[\text{cartesian}]{} \underline{\underline{\text{Set}}}$$

$$A \longrightarrow B^\circ * \mathbb{C} \simeq \mathcal{F}(B^\circ) \otimes \mathbb{C}$$

$$\underline{\underline{\text{FreeCartDist}}} \hookrightarrow \underline{\underline{\text{CartDist}}} : \underset{\text{small cat.}}{\mathbb{I}} \longmapsto \langle \mathbb{I} \rangle \overset{\text{def}}{=} \mathcal{F}(\mathbb{I})$$

$\underline{\underline{\mathrm{FreeCartDist}}} \hookrightarrow \underline{\underline{\mathrm{CartDist}}}$: $\underset{\text{small cat.}}{\mathbb{I}} \longmapsto \langle \mathbb{I} \rangle \overset{\mathrm{def}}{=} \mathcal{F}(\mathbb{I})$

$$\frac{\dfrac{\langle \mathbb{I} \rangle \;\leftarrow\!\!\!\!\!\rightarrow\; \langle \mathbb{J} \rangle \qquad \text{in } \underline{\underline{\mathrm{CartDist}}}}{(\mathcal{F}\mathbb{I})^{o} \times \mathcal{F}(\mathbb{J}) \xrightarrow[\text{right cart.}]{} \underline{\underline{\mathrm{Set}}}}}{\dfrac{(\mathcal{F}\mathbb{I})^{o} \times \mathbb{J} \longrightarrow \underline{\underline{\mathrm{Set}}}}{\mathcal{F}(\mathbb{I}) \;\longrightarrow\!\!\!\!\!\!+\;\; \mathbb{J} \qquad \text{in } \underline{\underline{\mathrm{Dist}}}}}$$

Enveloping the bicategorical Scott model

$$\underline{\underline{\mathrm{FreeCartDist}}} \hookrightarrow \underline{\underline{\mathrm{CartDist}}} : \underset{\text{small cat.}}{\mathbb{I}} \longmapsto \langle \mathbb{I} \rangle \overset{\mathrm{def}}{=} \mathcal{F}(\mathbb{I})$$

$$\frac{\dfrac{\dfrac{\langle \mathbb{I} \rangle \xrightarrow{\quad} \langle \mathbb{J} \rangle \quad \text{in } \underline{\underline{\mathrm{CartDist}}}}{(\mathcal{F}\mathbb{I})^{\circ} \times \mathcal{F}(\mathbb{J}) \xrightarrow[\text{right cart.}]{} \underline{\underline{\mathrm{Set}}}}}{(\mathcal{F}\mathbb{I})^{\circ} \times \mathbb{J} \longrightarrow \underline{\underline{\mathrm{Set}}}}}{\mathcal{F}(\mathbb{I}) \xrightarrow{\quad} \mathbb{J} \quad \text{in } \underline{\underline{\mathrm{Dist}}}}$$

$\simeq \underline{\mathrm{CoKl}}(\mathcal{F})$ [Cattani & Winskel (2005), Galal (2020), Olimpieri (2021)]

Cartesian Closure in the bicategorical Scott Model

- $\mathbb{1} \simeq \mathcal{F}(\mathbb{0}) = \langle 0 \rangle$
- $\langle \mathbb{I} \rangle \times \langle \mathbb{J} \rangle = \mathcal{F}(\mathbb{I}) \times \mathcal{F}(\mathbb{J}) \simeq \mathcal{F}(\mathbb{I} + \mathbb{J}) = \langle \mathbb{I} + \mathbb{J} \rangle$

Cartesian Closure in the bicategorical Scott Model

- $\mathbb{1} \simeq \mathcal{F}(\mathbb{0}) = \langle 0 \rangle$
- $\langle \mathbb{I} \rangle \times \langle \mathbb{J} \rangle = \mathcal{F}(\mathbb{I}) \times \mathcal{F}(\mathbb{J}) \simeq \mathcal{F}(\mathbb{I} + \mathbb{J}) = \langle \mathbb{I} + \mathbb{J} \rangle$
- $$\begin{aligned} \langle \mathbb{I} \rangle^{\circ} * \langle \mathbb{J} \rangle &= (\mathcal{F}\mathbb{I})^{\circ} * \mathcal{F}(\mathbb{J}) \\ &\simeq \mathcal{F}\big((\mathcal{F}\mathbb{I})^{\circ} \times \mathbb{J}\big) \\ &= \langle (\mathcal{F}\mathbb{I})^{\circ} \times \mathbb{J} \rangle \end{aligned}$$

An Abstract View

$$\begin{tikzcd} \underline{\underline{\mathbf{Cat}}} \arrow[loop left, "F"] \arrow[loop right, "P"] \end{tikzcd}$$

monad

relative monad

distributive law

[Cattani & Winskel (2005), Hyland (2010)]

An Abstract View

comonad $\mathcal{F}^{\bullet} \circlearrowleft \underline{\underline{\mathrm{Dist}}} \simeq (\underline{\underline{\mathrm{Cat}}}_{\mathcal{P}})^{op}$

$\mathcal{P}$-linearization

monad $\mathcal{F} \circlearrowleft \underline{\underline{\mathrm{Cat}}} \circlearrowright \mathcal{P}$ relative monad

distributive law

An Abstract View

Linear Logic

$\underline{\underline{\mathrm{Dist}}}_{\mathcal{F}^\bullet}$

cartesian closure

[Girard]

$\mathcal{F}^\bullet$

comonad $\underline{\underline{\mathrm{Dist}}} \simeq (\underline{\underline{\mathrm{Cat}}}_{\mathcal{P}})^{op}$

$\mathcal{P}$-linearization

$\underline{\underline{\mathrm{Cat}}}$

$\mathcal{F}$ $\mathcal{P}$

monad relative monad

distributive law

An Abstract View

Linear Logic

$\underline{\underline{\mathrm{Dist}}}_{\mathcal{F}^{\bullet}}$

cartesian closure

[Girard]

$\mathcal{P}$ relative monad

$\mathcal{F}^{\bullet}$ comonad $\underline{\underline{\mathrm{Dist}}} \simeq (\underline{\underline{\mathrm{Cat}}}_{\mathcal{P}})^{op}$

$\underline{\underline{\mathrm{Cat}}}^{\mathcal{F}} \simeq \underline{\underline{\mathrm{AlgTh}}}$

$\mathcal{P}$-linearization

$\mathcal{F}$-linearization

$\underline{\underline{\mathrm{Cat}}}$

$\mathcal{F}$ monad

$\mathcal{P}$ relative monad

distributive law

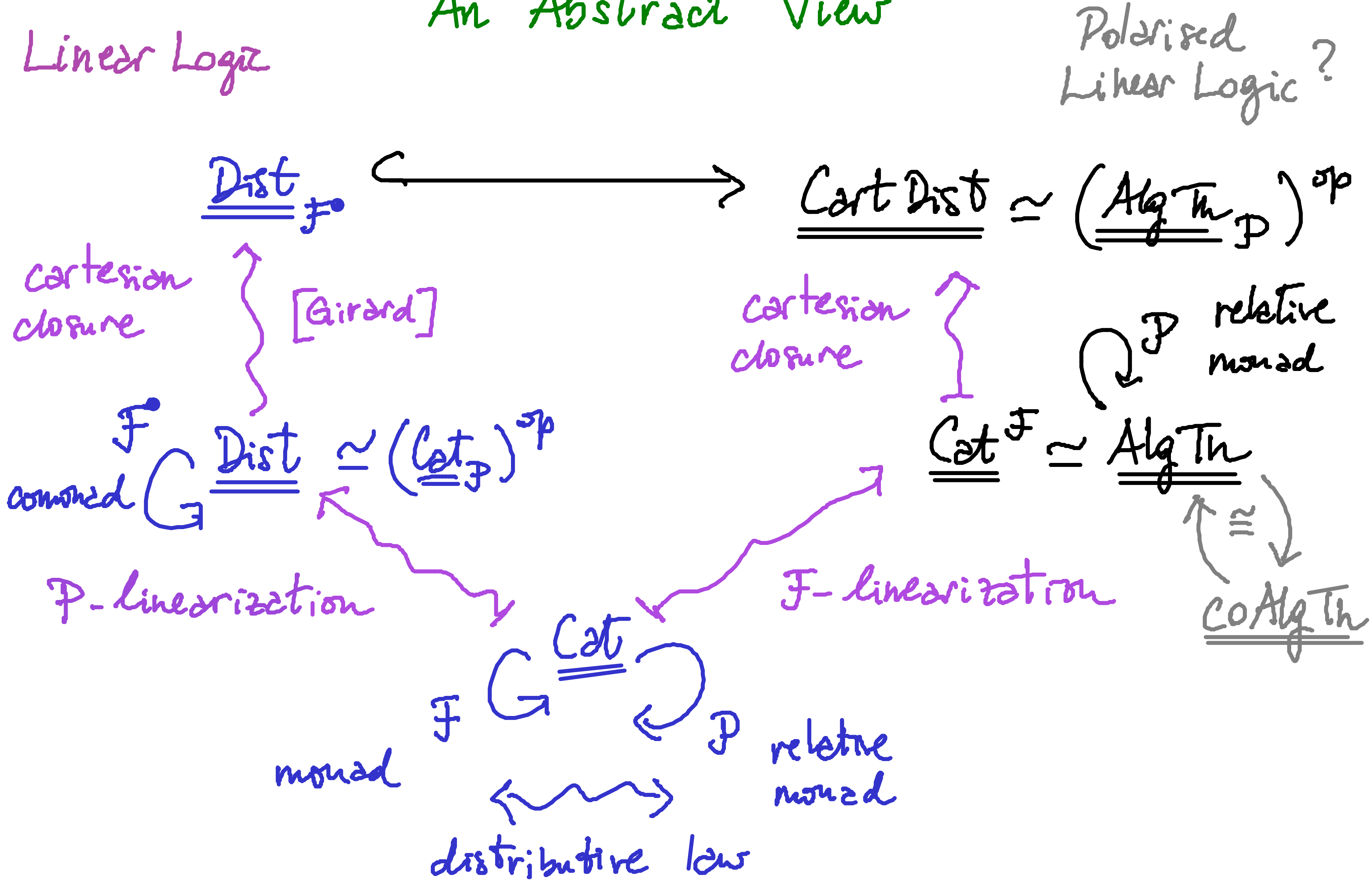
An Abstract View

Linear Logic

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathbf{Dist}_{\mathcal{F}^\bullet} \arrow[rr, hook] & & \mathbf{CartDist} \simeq (\mathbf{AlgTh}_{\mathcal{P}})^{op} \\
\mathbf{Dist} \simeq (\mathbf{Cat}_{\mathcal{P}})^{op} \arrow[u, rightsquigarrow, "\text{cartesian closure}", "{[\text{Girard}]}"'] \arrow[loop left, "{\mathcal{F}^\bullet \ \text{comonad}}"] & & \mathbf{Cat}^{\mathcal{F}} \simeq \mathbf{AlgTh} \arrow[u, rightsquigarrow, "\text{cartesian closure}"] \arrow[loop right, "{\mathcal{P} \ \text{relative monad}}"] \\
& \mathbf{Cat} \arrow[ul, leftrightsquigarrow, "\mathcal{P}\text{-linearization}"] \arrow[ur, leftrightsquigarrow, "\mathcal{F}\text{-linearization}"'] \arrow[loop left, "{\mathcal{F} \ \text{monad}}"] \arrow[loop right, "{\mathcal{P} \ \text{relative monad}}"] &
\end{tikzcd}$$

distributive law

An Abstract View



Two perspectives on cartesian distributors

$$\frac{\mathbb{A} \overset{C}{\longrightarrow} \mathbb{B}}{\mathbb{A}^{o} \times \mathbb{B} \xrightarrow[\text{right cartesian}]{} \underline{\underline{Set}}}$$

(I)

$$\mathbb{A}^{o} \longrightarrow \underline{\underline{CART}}(\mathbb{B}, \underline{\underline{Set}}) \overset{def}{=} \underline{Mod}(\mathbb{B})$$

algebraic models

(II)

$$\mathbb{B} \xrightarrow{\text{cartesian}} \underline{\underline{Set}}^{\mathbb{A}^{o}} \overset{def}{=} \mathcal{P}(\mathbb{A})$$

formal algebras-spaces duality

ALGEBRAIC PERSPECTIVE (I): Algebraic Theories and Models

- The category of models of an algebraic theory $\mathbb{A}$ in a cartesian category $\mathcal{S}$ is

$$\underline{\mathrm{Mod}}_{\mathcal{S}}(\mathbb{A}) =^{\mathrm{def}} \underline{\underline{\mathrm{CART}}}(\mathbb{A}, \mathcal{S})$$

__NB__:

$$\underline{\underline{\mathrm{CartDist}}}(\mathbb{A}, \mathbb{B}) \cong \underline{\mathrm{Mod}}_{\mathcal{P}(\mathbb{A})}(\mathbb{B})$$

$$\cong \underline{\mathrm{Mod}}(\mathbb{A}^{\circ} * \mathbb{B})$$

The Models Relative Monad

- $\underline{\underline{\mathrm{CAT}}}(\mathbb{A}^\circ, \mathcal{C}) \simeq \underline{\underline{\mathrm{sif}\,\mathrm{COCONT}}}(\underline{\mathrm{Mod}}(\mathbb{A}), \mathcal{C})$

For an algebraic theory $\mathbb{A}$,

$$\mathbb{A}^\circ \hookrightarrow \underline{\mathrm{Mod}}(\mathbb{A}) : a \mapsto \mathbb{A}(a,-)$$

is the sifted cocompletion of $\mathbb{A}^\circ$

The Models Relative Monad

- $\underline{\underline{\mathrm{CAT}}}(\mathbb{A}^\circ, \mathcal{C}) \simeq \underline{\underline{\mathrm{sif\,COCONT}}}(\underline{\mathrm{Mod}}(\mathbb{A}), \mathcal{C})$

  For an algebraic theory $\mathbb{A}$,

  $$\mathbb{A}^\circ \hookrightarrow \underline{\mathrm{Mod}}(\mathbb{A}) : a \mapsto \mathbb{A}(a,-)$$

  is the sifted cocompletion of $\mathbb{A}^\circ$

- $\underline{\underline{\mathrm{CartDist}}} \simeq \underline{\mathrm{Kl}}(\underline{\mathrm{Mod}}^\bullet)$

  for $\underline{\mathrm{Mod}}^\bullet : \underline{\underline{\mathrm{coAlgTh}}} \longrightarrow \underline{\underline{\mathrm{sif\,COCONT}}} : \mathbb{C} \longmapsto \underline{\mathrm{Mod}}(\mathbb{C}^\circ)$

Domain-theoretic Perspective

$$\underline{\underline{\mathbf{CartDist}}} \simeq \begin{cases} \text{objects: algebraic theories} \\ \text{morphisms: functors between categories} \\ \qquad\qquad\quad \text{of models preserving} \\ \qquad\qquad\quad \text{filtered colimits and} \\ \qquad\qquad\quad \text{reflexive coequalizers} \\ \text{cells: natural transformations} \end{cases}$$

ALGEBRAIC PERSPECTIVE (II): Enveloping rigs and spaces

$$\underline{\underline{\mathrm{CART}}}(\mathbb{A}, \mathscr{C}) \simeq \underline{\underline{\mathrm{CART/COCONT}}}(\mathcal{P}(\mathbb{A}), \mathscr{C})$$

For an algebraic theory $\mathbb{A}$, the Yoneda embedding

$$\mathbb{A} \hookrightarrow \mathcal{P}(\mathbb{A}) : a \mapsto \mathbb{A}(-, a)$$

exhibits $\mathcal{P}(\mathbb{A})$ as the free cartesianly cocomplete category on $\mathbb{A}$

cocomplete categories with left-and-right cocontinuous finite products

- $\underline{\underline{\mathsf{CartDist}}} \simeq \underline{\mathsf{KL}}(\mathcal{P})^{op}$

$$
\begin{tikzcd}
\mathsf{Cart} \arrow[rr, "\mathcal{P}"] \arrow[dr, dashed] & & \underline{\underline{\mathsf{CART/COCONT}}} \\
& \underline{\underline{\mathsf{CART2RIGS}}} \arrow[ur] &
\end{tikzcd}
$$

(objects: locally presentable cartesian categories)

- $\underline{\underline{\mathrm{CartDist}}} \simeq \underline{KL}(\mathcal{P})^{op}$

\begin{tikzcd}
\mathrm{Cart} \arrow[rr, "\mathcal{P}"] \arrow[dr, dashed] & & \underline{\underline{\mathrm{CART/COCONT}}} \\
& \underline{\underline{\mathrm{CART2RIGS}}} \arrow[ur] &
\end{tikzcd}

(objects: locally presentable cartesian categories)

- $\underline{\underline{\mathrm{CartDist}}} \overset{\text{freely}}{\hookrightarrow} \underline{\underline{\mathrm{CART2RIGS}}}^{op} =^{\mathrm{def}} \underline{\underline{\mathrm{ParaTop}}}$

generalised spaces

- $\underline{\underline{\mathrm{CartDist}}} \simeq \underline{\mathrm{Kl}}(\mathcal{P})^{op}$

$$
\begin{tikzcd}
\mathrm{Cart} \arrow[rr, "\mathcal{P}"] \arrow[dr, dashed] & & \underline{\underline{\mathrm{CART/COCONT}}} \\
& \underline{\underline{\mathrm{CART2RIGS}}} \arrow[ur, hook] &
\end{tikzcd}
$$

(objects: locally presentable cartesian categories)

- $\underline{\underline{\mathrm{CartDist}}} \xhookrightarrow{\text{freely}} \underline{\underline{\mathrm{CART2RIGS}}}^{op} =^{def} \underline{\underline{\mathrm{ParaTop}}}$ generalised spaces

$$
\begin{array}{ccc}
\wr & & \wr \\
\text{cartesian} & \Longleftarrow & \text{cartesian}
\end{array}
$$

- $\mathrm{CartDist} \simeq \mathrm{KL}(\mathcal{P})^{op}$

$$\begin{tikzcd}
\mathrm{Cart} \arrow[r, "\mathcal{P}"] \arrow[d, dashed] & \mathrm{CART/COCONT} \\
\mathrm{CART2RIGS} \arrow[ur] &
\end{tikzcd}$$

(objects: locally presentable cartesian categories)

- $\mathrm{CartDist} \xhookrightarrow{\text{freely}} \mathrm{CART2RIGS}^{op} =^{\mathrm{def}} \mathrm{ParaTop}$

generalised spaces

cartesian	$\Longleftarrow$		cartesian
closed	$\Longleftarrow$	free paratoposes on alg. theories are exponentiable	$\Longleftarrow$ has powers and tensors

RESEARCH THEMES

- Models over

  TOKENS WITH STRUCTURE

- Approaches to

  LINEARIZATION

- Underlying

  LOGICAL SYSTEM

References

[Cattani & Winskel (2005)] Profunctors, Open Maps and Bisimulation. MSCS.

[Fiore & Joyal (2020)] Tensors of KZ-algebras. ALT (CMUC) seminar.

[Fiore & Gambino & Hyland & Winskel (2018)] Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures. Selecta Mathematica.

[Fiore & Joyal (2015)] Theory of para-toposes. CT talk.

[Fiore & Joyal (2021)] A cartesian closed category of algebraic theories. MCA talk.

[Galal (2020)] A profunctorial Scott Semantics. FSCD.

[Hyland (2010)] Some reasons for generalising domain theory. MSCS.

[Olimpieri (2021)] Intersection type distributors. LICS.